P.N.Lebedev Institute preprint FIAN/TD-15/92

I.E.Tamm Theory Department hepth@xxx/92##

September 1992

Topological versus Non–Topological Theories and $p - q$ Duality in $c \leq 1$ 2d Gravity Models

S. Kharchev, A. Marshakov [1]

P.N.Lebedev Physics Institute

Leninsky prospect, 53, Moscow, 117 924, Russia, [2]

Abstract

We discuss the non–perturbative formulation for $c \leq 1$ string theory. The field theory like formulation of topological and non–topological models is presented. The integral representation for arbitrary (p, q) solutions is derived which explicitly obeys $p - q$ duality of these theories. The exact solutions to string equation and various examples are also discussed.

[1] talk given by A.M. at the International Workshop on String Theory, Quantum Gravity and the Unification of Fundamental Interactions, Rome, 21 – 26 September 1992.

[2] E-mail address: tdparticle@glas.apc.org

1 Introduction

Recent years brought us to a great progress in understanding of non-perturbative string theory. The key idea, established at least for the most simple set of $c \leq 1$ conformal theories interacting with two-dimensional gravity, is the appearance of the structure of integrable hierarchy in the description of generating function for physical correlators in these models [1, 2].

Fortunately, the particular solutions to non-perturbative string theory can be singled from the whole set of solutions to the Kadomtsev-Petviashvili (KP) or rather Toda lattice hierarchy by an additional requirement usually known in the form of string equation, which allows to present these particular solutions in a conventional "field-theory-like" form. Indeed, it serious of papers [3, 4, 5, 6, 7] it was shown, that there exists a certain matrix model, describing the particular subset of solutions to (reduced) KP-hierarchy satisfying at the same time the string equation. The proposed matrix theory can be considered as unifying theory for $c \leq 1$ coupled to 2d gravity string models, allowing one to interpolate among them [3, 4, 6], thus being a sort of effective string field theory [7].

Below, we are going to investigate solutions to various (p, q)-models (with central charges $c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$) coupled to $2d$ gravity in more details. Moreover, we would stress the advantages of matrix integral (or better multiple integral [1]) representation for the particular "stringy" solutions to KP hierarchy.

The basic feature of these matrix integrals is that they give an explicit solution to string equation around a *topological* point. By definition, for the simplest case of $(p = 2, q = 1)$ theory such integral was derived by Kontsevich [8, 9] when studying topological characteristics of 2d gravity module space. In its basic form it obeys all the properties of topological theory – trivial continuum limit (the size of matrix $N \to \infty$) means actually N-independence while N can be interpreted as a cutoff parameter, very simple form of particular solutions (Airy function etc), deep interrelation with the Landau-Ginzburg topological theories and good quasiclassical properties [6]. However, the problem of higher critical points is much more complicated question.

Below, we are going to argue, that it also gets much better understanding along this line. Shortly, higher critical points can be described using the same "action" principle, based on study of the quasi-classical limit [10, 11], but the exact answer has much more complicated form. It means that the higher critical points are no more *topological* theories in the naive sense we used above. [2] The integral representation for these solutions besides the "action" functional has very complicated structure of integration measure. Nevertheless, this integral representation obeys the basic property of $p - q$ duality in the spirit of [12] and might turn to be useful for studying the exact solutions in various cases.

In sect.2 we are going to repeat the main results of [6] on topological solutions and speculate on naive

[1] Indeed, the naive very simple matrix form for such solutions might not be really their basic feature. For example, in the case of ordinary discrete matrix models the generalization to "multimatrix case" is better done via *conformal multimatrix models* which do not have an elegant representation by *matrix* integral, but are *multiple* integral solutions to the extended (discrete) Virasoro-W constraints [13].

[2] Of course, they could still be reminisents of some more complicated topological theories (and their module spases) like W-gravity etc (see for example [26] and J.-L.Gervais's contribution to this volume).

"analytic continuation" to higher critical points. In sect.3 we will formulate the "action" principle and derive an integral representation for arbitrary (p,q)-solutions. In sect.4 we consider $p-q$ symmetry and present several simple and useful examples. Sect.5 contains several examples of $c<1$ exact (p,q)-solutions and sect.6 – some comments on what is supposed to be a particular example of $c=1$ situation. In sect.7 we give several concluding remarks.

2 Topological $(p,1)$ models

First, we remind that the partition function is defined [3, 4] as a matrix integral

$$Z^{(N)}[V|M] \equiv C^{(N)}[V|M]e^{TrV(M)-TrMV'(M)}\int DX\ e^{-TrV(X)+TrV'(M)X} \tag{1}$$

over $N\times N$ "Hermitean" matrices, with the normalizing factor given by Gaussian integral

$$C^{(N)}[V|M]^{-1} \equiv \int \mathrm{DY}\ \ e^{-TrV_2[M,Y]},$$

$$V_2 \equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}Tr[V(M+\epsilon Y)-V(M)-\epsilon YV'(M)] \tag{2}$$

and Z actually depends on M only through the invariant variables

$$T_k = \frac{1}{k}Tr\ M^{-k},\ k\geq 1\ ; \tag{3}$$

moreover, if rewritten in terms of T_k, $Z[V|T] = Z^{(N)}[V|M]$ is actually independent of the size N of the matrices.

As a function of T_k $Z[V|T_k]$ is a τ-function of KP-hierarchy, $Z[V|T_k] = \tau_V[T_k]$, while the potential V specifies (up to certain invariance) the relevant point of the infinite-dimensional Grassmannian.

For various choices of the potential $V(X)$ the model (1) formally reproduces various (p,q)-series: the potential $V(X) = \frac{X^{p+1}}{p+1}$ can be associated with the entire set of (p,q)-minimal string models with all possible q's. In order to specify q one needs to make a special choice of T-variables: all $T_k = 0$, except for T_1 and T_{p+q} (the symmetry between p and q is implicit in this formulation).

However, this is only a formal consideration. For the potential $V(X) = \frac{X^{p+1}}{p+1}$ the partition function $Z[V|T_k] = \tau_V[T_k] \equiv \tau_p[T_k]$ satisfies the string equation which looks like

$$\sum_{k=1}^{p-1}k(p-k)T_kT_{p-k} + \sum_{k=1}^{\infty}(p+k)(T_{p+k} - \frac{p}{p+1}\delta_{k,1})\frac{\partial}{\partial T_k}\log\ \tau_p[T] = 0 \tag{4}$$

i.e. τ-function is defined with all Miwa times (3) around zero values (in $1/M$ decomposition like in original Kontsevich model) with the only exception - T_{p+1} is shifted what corresponds obviously to $(p,1)$ model. Thus, we see that the matrix integral gives an explicit solution to $(p,1)$ string models which should have mentioned above topological properties and must be nothing but particular topological matter coupled to topological gravity.

Of course, we still have an opportunity for analytic continuation in string equation, using the definition of Miwa's times (3). We have to satisfy the following conditions:

$$T_1 = x$$

$$T_2 = 0$$

$$\dots$$

$$T_{p+1} - \frac{p}{p+1} = 0$$

$$T_{p+q} = t_{p+q} = fixed$$

$$T_{p+q+1} = 0$$

$$\dots \tag{5}$$

which is a system of equations on the Miwa parameters $\{\mu_i\}$, $i = 1, ..., N$. So, to do this analytical continuation one has to decompose the whole set

$$\{\mu_i\} = \{\xi_a\} \oplus \{\mu'_s\}$$

$$T_k = \frac{1}{k} TrM^{-k} = \frac{1}{k}\sum_{j=1}^{N} \mu_j^{-k} = \frac{1}{k}\sum \xi_a^{-k} + \frac{1}{k}\sum_{j=1}^{N'} {\mu'}_j^{-k} \equiv T_k^{(cl)} + T'_k \tag{6}$$

into "classical" and "quantum" parts respectively. In principle it is clear that we have now to solve the equations

$$T_k^{(cl)} = \frac{1}{k}\sum \xi_a^{-k} = t_{p+q}\delta_{k,p+q} - \frac{p}{p+1}\delta_{k,p+1} \tag{7}$$

and this can be done adjusting a certain block form of the matrix M [4, 7]. However, in such a way we can only vanish several first times, and the rest ones can be vanished only adjusting correct behaviour in the limit $N \to \infty$. The most elegant way to do this [3] is to look at the formula

$$\exp(-\sum_{k=1}^{\infty} \lambda^k T_k^{(cl)}) = \lim_{K\to\infty} (1 - \frac{1}{K}\sum_{k=1}^{\infty} \lambda^k T_k^{(cl)})^K = \prod_a (1 - \frac{\lambda}{\xi_a}) \tag{8}$$

and then the solution to (7) will be given by K sets of roots of the equation

$$\sum_{k=1}^{\infty} \lambda^k T_k^{(cl)} - K = t_{p+q}\lambda^{p+q} - \frac{p}{p+1}\lambda^{p+1} - K = 0 \tag{9}$$

Obviously, the eigenvalues ξ_a will now depend on the size of the matrix $N = (p+q)K + N'$ through explicit K-dependence ($\xi_a \sim K^{1/(p+q)}$) and we lose one of the main features of $(p, 1)$ theories mentioned above — trivial dependence of the size of the matrix. Now we can consider only matrices of *infinite* size and deal only with the infinite determinant formulas.

That is why we call such way to get higher critical points as a formal one. Below we will try to understand an alternative way of thinking, connected with so-called p-times. Indeed, it was noticed in [6]

[3] suggested by A.Zabrodin

that there exists *a priori* another integrable structure in the model (1), connected with time variables, related to the non-trivial coefficients of the potential V. As a results, the cases of monomial potential $V_p(X) = \frac{X^{p+1}}{p+1}$ and arbitrary polynomial of the same degree $(p+1)$ are closely connected with each other. The direct calculation shows (see [6] for details)

$$Z[V|T_k] = \tau_V[T_k] =$$
$$= \exp\left(-\frac{1}{2}\sum A_{ij}(t)(\tilde{T}_i - t_i)(\tilde{T}_j - t_j)\right)\tau_p[\tilde{T}_k - t_k]\ , \tag{10}$$

where

$$V(x) = \sum_{i=0}^{p}\frac{v_i}{i}x^i$$
$$\tilde{T}_k = \frac{1}{k}Tr\tilde{M}^{-k}\ ,$$
$$\tilde{M}^p = V'(M) \equiv W(M)\ ,$$
$$A_{ij} = Res_\mu W^{i/p}dW_+^{j/p}\ , \tag{11}$$

where $f(\mu)_+$ denotes the positive part of the Laurent series $f(\mu) = \Sigma\ f_i\mu^i$ and

$$\tau_p[T] \equiv \tau_{V_p}[T] \tag{12}$$

– is the τ-function of p-reduction. The parameters $\{t_k\}$ are certain linear combinations of the coefficients $\{v_k\}$ of the potential [17, 10]

$$t_k = -\frac{p}{k(p-k)}Res\ W^{1-k/p}(\mu)d\mu \tag{13}$$

Formula (10) means that "shifted" by flows along p-times (13) τ-function is easily expressed through the τ-function of p-reduction, depending only on the difference of the time-variables $\tilde{T}_k$ and t_k. The change of the spectral parameter in (5) $M \to \tilde{M}$ (and corresponding transformation of times $T_k \to \tilde{T}_k$) is a natural step from the point of view of equivalent hierarchies.

The τ-functions in (10) are defined by formulas

$$\tau_V[T] = \frac{\det\ \phi_i(\mu_j)}{\Delta(\mu)} \tag{14}$$

and

$$\frac{\tau_p[\tilde{T} - t]}{\tau_p[t]} = \frac{\det\hat{\phi}_i(\tilde{\mu}_j)}{\Delta(\tilde{\mu})} \tag{15}$$

with the corresponding points of the Grassmannian determined by the basic vectors

$$\phi_i(\mu) = [W'(\mu)]^{1/2}\exp\left(V(\mu) - \mu W(\mu)\right)\int x^{i-1}e^{-V(x)+xW(\mu)}dx \tag{16}$$

and

$$\hat{\phi}_i(\tilde{\mu}) = [p\tilde{\mu}^{p-1}]^{1/2}\exp\left(-\sum_{j=1}^{p+1}t_j\tilde{\mu}_j\right)\int x^{i-1}e^{-V(x)+x\tilde{\mu}^p}dx \tag{17}$$

respectively. Then it is easy to show that $\hat{\tau}_p(T)$ satisfies the L_{-1}- constraint with *shifted* KP-times in the following way

$$\sum_{k=1}^{p-1} k(p-k)(\tilde{T}_k - t_k)(\tilde{T}_{p-k} - t_{p-k}) + \sum_{k=1}^{\infty}(p+k)(\tilde{T}_{p+k} - t_{p+k})\frac{\partial}{\partial \tilde{T}_k} \log \hat{\tau}_p[\tilde{T} - t] = 0 \tag{18}$$

where t_i defined by (13) are *identically* equal to zero for $i \geq p+2$.

The formulas (10,18) demonstrate at least two things. First, the partition function in the case of deformed monomial potential ($\equiv$ *polynomial* of the same degree) is expressed through the equivalent solution (in the sense [14, 15]) of the same p-reduced KP hierarchy, second – not only t_{p+1} but all t_k with $k \leq p+1$ are not equal to zero in the deformed situation. We will call such theories as *topologically deformed* $(p,1)$ models (in contrast to *pure* $(p,1)$ models given by monomial potentials $V_p(X)$), the deformation is "topological" in the sense that it preserves all the features of topological models we discussed above. Moreover, this "topological" deformation preserves almost all features of $2d$ Landau-Ginzburg theories and from the point of view of continuum theory they should be identified with the twisted Landau-Ginzburg topological matter interacting with gravity.

From the point of view of KP hierarchy we deal now again with p-reduction. Indeed, from eq.(15) in the limit when all of the Miwa variables $\tilde{\mu}_i$ go to infinity except of the first eigenvalue ($\tilde{\mu}_i \equiv \tilde{\mu}$) one can obtain the expression for the Baker-Akhiezer function which is almost equal to the first basic vector:

$$\psi(\tilde{\mu}, t) = \exp\left(\sum_{j=1}^{p+1} t_j \tilde{\mu}_j\right) \hat{\phi}_1(\tilde{\mu}) = [p\tilde{\mu}^{p-1}]^{1/2} \int e^{-V(x)+x\tilde{\mu}^p} dx \tag{19}$$

where potential $V(x)$ is parameterized by p-times $\{t_k\}$ due to eq.(13). It is evident that $\psi(\tilde{\mu}, t)$ has the usual asymptotic

$$\psi(\tilde{\mu}, t) \underset{\mu\to\infty}{\to} \exp\left(\sum_{j=1}^{p+1} t_j \tilde{\mu}_j\right) \left(1 + O(\tilde{\mu}^{-1})\right) \tag{20}$$

Using equations of motion for *quasiclassical* KP hierarchy [17, 10]

$$\frac{\partial V}{\partial t_i} = -W_+^{i/p} \tag{21}$$

(this is the consequence of parameterization (13)) one can easy to show that the Baker-Akhiezer function (19) satisfies the usual equations of the p-reduced KP hierarchy:

$$[W(\frac{\partial}{\partial t_1}) + t_1]\psi(\tilde{\mu}, t) = \tilde{\mu}^p \psi(\tilde{\mu}, t) \ ,$$

$$\frac{\partial \psi}{\partial t_i} = W_+^{i/p}(\frac{\partial}{\partial t_1})\psi \ . \tag{22}$$

where polynomials $W_+^{i/p}(\mu)$ are functions of p-times (13). It is important that $W_+^{i/p}(\mu)$ does not depend on t_1 for $i < p$ and, therefore, in the corresponding Zakharov-Shabat equations we can treat $\partial/\partial t_1$ as a formal *parameter*, not an operator. Thus, we see that topologically deformed $(p,1)$ models which are quasiclassical limit of the KP hierarchy in the sense of [10] are simultaneously the *exact* solutions of the full p-KP hierarchy restricted on the "small phase space" [16]. The Baker-Akhiezer function (19) represent the explicit solution of evolution equations along first p flows and all basic vectors of the deformed $(p,1)$ model can be obtained from $\psi(\tilde{\mu},t)$ with the help of the formula

$$\phi_i(\tilde{\mu},t) = \exp\left(-\sum_{j=1}^{p+1} t_j \tilde{\mu}_j\right) \frac{\partial^{i-1}\psi(\tilde{\mu},t)}{\partial t_1^{i-1}} \tag{23}$$

(Of course, the basic vectors of the pure $(p,1)$ model corresponding to monomial potential V_p can be obtained by setting $t_1 = ... = t_p = 0$, $t_{p+1} = \frac{p}{p+1}$). As a solution to string equation this deformed case differs only in analytic continuation along first p times.

These topologically deformed $(p,1)$ models as we already said preserve all topological properties of $(p,1)$ models. Indeed, according to [2] shifting of first times $t_1, ..., t_{p+1}$ is certainly not enough to get higher critical points. To do this one has to obtain $t_{p+q} \neq 0$, but this cannot be done using above formulas naively, because it is easily seen from definition (13) of p-times, that $t_k \equiv 0$ for $k \geq p+2$. To do this we have to modify the above procedure and we are going to this in next section.

3 Action principle

The above scheme has a natural quasiclassical interpretation. Indeed, the solution to $(p,1)$ theories given by the partition function (1) can be considered as a "path integral" representation of the solution to Douglas equations [1]

$$[\hat{P},\hat{Q}] = 1 \tag{24}$$

where $\hat{P}$ and $\hat{Q}$ are certain differential operators (of order p and q) respectively and obviously $p-th$ order of $\hat{P}$ dictates p-reduction, while q stands for $q-th$ critical point. Quasiclassically, (24) turns into Poisson brackets relation [10, 11]

$$\{P,Q\} = 1 \tag{25}$$

where $P(x)$ and $Q(x)$ are now certain (polynomial) functions. It is easily seen that the above case corresponds to the first order polynomial $Q(x) \equiv x$ and the p-th order polynomial $P(x)$ should be identified with $W(x) \equiv V'(x)$ [10]. Thus, the exponentials in (1), (16) and (17) acquire an obvious sense of action functionals

$$\begin{gathered}
S_{p,1}(x,\mu) = -V(x) + xW(\mu) = -\int_0^x dy\ W(y)Q'(y) + Q(x)W(\mu) \\
W(x) = V'(x) = x^p + \sum_{k=1}^{p} v_k x^{k-1} \\
Q(x) = x
\end{gathered} \tag{26}$$

and we claim that the generalization to arbitrary (p,q) case must be

$$\begin{gathered}
S_{W,Q} = -\int_0^x dy\ W(y)Q'(y) + Q(x)W(\mu) \\
W(x) = V'(x) = x^p + \sum_{k=1}^{p} v_k x^{k-1}
\end{gathered}$$

$$Q(x) = x^q + \sum_{k=1}^{q} \bar{v}_k x^{k-1} \tag{27}$$

Now the "true" co-ordinate is Q, therefore the extreme condition of action (27) is still

$$W(x) = W(\mu) \tag{28}$$

having $x = \mu$ as a solution, and for extreme value of the action one gets

$$S_{W,Q}|_{x=\mu} = \int_0^\mu dy\ W'(y)Q(y) =$$

$$= \sum_{k=-\infty}^{p+q} t_k \tilde{\mu}^k \tag{29}$$

where $\tilde{\mu}^p = W(\mu)$ and

$$t_k \equiv t_k^{(W,Q)} = -\frac{p}{k(p-k)} Res\ W^{1-k/p} dQ\ . \tag{30}$$

We should stress that the extreme value of the action (27), represented in the form (29), determines the quasiclassical (or dispersionless) limit of the p-reduced KP hierarchy [10, 11] with $p+q-1$ independent flows. We have seen that in the case of topologically deformed $(p,1)$ models the quasiclassical hierarchy is exact in the strict sense: topological solutions satisfy the full KP equations and the first basic vector is just the Baker-Akhiezer function of our model (1) restricted to the small phase space. Unfortunately, this is not the case for the general (p,q) models: now the quasiclassics is not exact and in order to find the basic vectors in the explicit form one should solve the original problem and find the exact solutions of the full KP hierarchy along first $p+q-1$ flows. Nevertheless, we argue that the presence of the "quasiclassical component" in the whole integrable structure of the given models is of importance and it can give, in principle, some useful information, for example, we can make a conjecture that the coefficients of the basic vectors are determined by the derivatives of the corresponding *quasiclassical* τ-function [4].

Returning to eq.(30) we immediately see, that now only for $k \geq p+q+1$ p-times are identically zero, while

$$t_{p+q} \equiv t_{p+q}^{(W,Q)} = \frac{p}{p+q} \tag{31}$$

and we should get a correct critical point adjusting all $\{t_k\}$ with $k < p+q$ to be zero. The exact formula for the Grassmannian basis vectors in general case acquires the form

$$\phi_i(\mu) = [W'(\mu)]^{1/2} \exp(-\ S_{W,Q}|_{x=\mu}) \int d\mathcal{M}_Q(x) f_i(x) \exp\ S_{W,Q}(x,\mu) \tag{32}$$

where $d\mathcal{M}_Q(x)$ is the integration measure. We are going to explain, that the integration measure for generic theory determined by two arbitrary polynomials W and Q has the form

$$d\mathcal{M}_Q(z) = [Q'(z)]^{1/2} dz \tag{33}$$

[4] The difference between generic (p,q) and $(p,1)$ cases is also crucial from the point of view of topological nature. We can see here again a distinction between what we call topological and naively non-topological models. The complications in general (p,q) case might be connected with the fact that we use not the most convenient representation for these theories (see also footnote on the second page)

by checking the string equation. For the choice (33) to insure the correct asymptotics of basis vectors $\phi_i(\mu)$ we have to take $f_i(x)$ being functions (not necessarily polynomials) with the asymptotics

$$f_i(x) \sim x^{i-1}(1 + O(1/x)) \tag{34}$$

4 p-reduction and the Kac-Schwarz operator

To satisfy the string equation, one has to fulfill two requirements: the reduction condition

$$W(\mu)\phi_i(\mu) = \sum_j C_{ij}\phi_j(\mu) \tag{35}$$

and the Kac-Schwarz [19, 20] operator action

$$A^{(W,Q)}\phi_i(\mu) = \sum A_{ij}\phi_j(\mu) \tag{36}$$

with

$$\begin{aligned} A^{(W,Q)} \equiv N^{(W,Q)}(\mu)\frac{1}{W'(\mu)}\frac{\partial}{\partial\mu}[N^{(W,Q)}(\mu)]^{-1} = \\ = \frac{1}{W'(\mu)}\frac{\partial}{\partial\mu} - \frac{1}{2}\frac{W''(\mu)}{W'(\mu)^2} + Q(\mu) \\ N^{(W,Q)}(\mu) \equiv [W'(\mu)]^{1/2}\exp(- \left.S_{W,Q}\right|_{x=\mu}) \end{aligned} \tag{37}$$

These two requirements are enough to prove string equation (see [4] for details). The structure of action immediately gives us that

$$A^{(W,Q)}\phi_i(\mu) = N^{(W,Q)}(\mu)\int d\mathcal{M}_Q(z)Q(z)f_i(z)\exp\ S_{W,Q}(z,\mu) \tag{38}$$

and the condition (36) can be reformulated as a Q-reduction property of basis $\{f_i(z)\}$

$$Q(z)f_i(z) = \sum A_{ij}f_i(z) \tag{39}$$

Let us check now the reduction condition. Multiplying $\phi_i(\mu)$ by $W(\mu)$ and integrating by parts we obtain

$$\begin{aligned} W(\mu)\phi_i(\mu) = \\ = N^{(W,Q)}(\mu)\int d\mathcal{M}_Q(z)f_i(z)\frac{1}{Q'(z)}\frac{\partial}{\partial z}[\exp\ Q(z)W(\mu)]\exp[-\int_0^z dy\ W(y)Q'(y)] = \\ = -N^{(W,Q)}(\mu)\int d\mathcal{M}_Q(z)\exp[S_{W,Q}(z,\mu)]\left(\frac{1}{Q'(z)}\frac{\partial}{\partial z} - \frac{1}{2}\frac{Q''(z)}{Q'(z)^2} - W(z)\right)f_i(z) \equiv \\ \equiv -N^{(W,Q)}(\mu)\int d\mathcal{M}_Q(z)\exp[S_{W,Q}(z,\mu)]A^{(Q,W)}f_i(z) \end{aligned} \tag{40}$$

Therefore, in the "dual" basis $\{f_i(z)\}$ the condition (31) turns to be

$$A^{(Q,W)}f_i(z) = -\sum C_{ij}f_j(z) \tag{41}$$

with $A^{(Q,W)}(\neq A^{(W,Q)})$ being the "dual" Kac-Schwarz operator

$$A^{(Q,W)} = \frac{1}{Q'(z)}\frac{\partial}{\partial z} - \frac{1}{2}\frac{Q''(z)}{Q'(z)^2} - W(z) \tag{42}$$

The representation (32), (33) is an exact integral formula for basis vectors solving the (p,q) string model. It has manifest property of $p-q$ duality (in general $W-Q$), turning the (p,q)-string equation into the equivalent (q,p)-string equation.

Now let us transform (32), (33) into a little bit more explicit $p-q$ form. As before for $(p,1)$ models we have to make substitutions, leading to equivalent KP solutions:

$$\tilde{\mu}^p = W(\mu),\ \tilde{z}^q = Q(z) \tag{43}$$

Then we can rewrite (32) as

$$\hat{\phi}_i(\tilde{\mu}) = [p\tilde{\mu}^{p-1}]^{1/2}\exp\left(-\sum_{k=1}^{p+q} t_k\tilde{\mu}^k\right)\int d\tilde{z}[q\tilde{z}^{q-1}]^{1/2}\hat{f}_i(\tilde{z})\exp\ S_{W,Q}(\tilde{z},\tilde{\mu}) \tag{44}$$

where action is given now by

$$\begin{aligned} S_{W,Q}(\tilde{z},\tilde{\mu}) &= -\left[\int_0^{\tilde{z}} d\tilde{y}q\tilde{y}^{q-1}W(y(\tilde{y}))\right]_+ + \tilde{z}^q\tilde{\mu}^p \\ &= \sum_{k=1}^{p+q}\bar{t}_k\tilde{z}^k + \tilde{z}^q\tilde{\mu}^p \end{aligned} \tag{45}$$

In new coordinates the reduction conditions are

$$\begin{aligned} \tilde{\mu}^p\hat{\phi}_i(\tilde{\mu}) &= \sum_j \tilde{C}_{ij}\hat{\phi}_j(\tilde{\mu}) \\ \tilde{z}^q\hat{f}_i(\tilde{z}) &= \sum_j \tilde{A}_{ij}\hat{f}_j(\tilde{z}) \end{aligned} \tag{46}$$

and for the Kac-Schwarz operators one gets conventional formulas [19, 20, 4]

$$\begin{aligned} \tilde{A}^{(p,q)} &= \frac{1}{p\tilde{\mu}^{p-1}}\frac{\partial}{\partial\tilde{\mu}} - \frac{p-1}{2p}\frac{1}{\tilde{\mu}^p} + \frac{1}{p}\sum_{k=1}^{p+q} kt_k\tilde{\mu}^{k-p} \\ \tilde{A}^{(q,p)} &= \frac{1}{q\tilde{z}^{q-1}}\frac{\partial}{\partial\tilde{z}} - \frac{q-1}{2q}\frac{1}{\tilde{z}^q} + \frac{1}{q}\sum_{k=1}^{p+q} k\bar{t}_k\tilde{z}^{k-q} \end{aligned} \tag{47}$$

where for (q,p) models we have introduced the "dual" times:

$$\bar{t}_k \equiv t_k^{(Q,W)} = \frac{q}{k(q-k)} Res\ Q^{1-k/q}dW \tag{48}$$

in particularly, $\bar{t}_{p+q} = -\frac{q}{p}t_{p+q} = -\frac{q}{p+q}$. Now string equations give correspondingly

$$\begin{aligned} \tilde{A}^{(p,q)}\hat{\phi}_i(\tilde{\mu}) &= \sum \tilde{A}_{ij}\hat{\phi}_j(\tilde{\mu}) \\ \tilde{A}^{(q,p)}\hat{f}_i(\tilde{z}) &= -\sum \tilde{C}_{ij}\hat{f}_j(\tilde{z}) \end{aligned} \tag{49}$$

By these formulas we get a manifestation of $p-q$ duality if solutions to $2d$ gravity.

5 Examples: topological and non-topological theories

Now, let us consider briefly several explicit examples. First, for monomials $W(x) = x^p$ and $Q(x) = x^q$, $\tilde{\mu} \equiv \mu$, $\tilde{z} \equiv z$, $\hat{\phi}_i \equiv \phi_i$ and $\hat{f}_i \equiv f_i$, thus, the formulas of the previous section will be

$$\phi_i(\mu) = [p\mu^{p-1}]^{1/2} \exp\left(-\frac{p}{p+q}\mu^{p+q}\right) \times$$
$$\times \int dz [qz^{q-1}]^{1/2} f_i(z) \exp\left(-\frac{q}{p+q}z^{p+q} + z^q\mu^p\right) \tag{50}$$

and the Kac-Schwarz operators acquire the most simple form

$$A^{(p,q)} = \frac{1}{p\mu^{p-1}}\frac{\partial}{\partial\mu} - \frac{p-1}{2p}\frac{1}{\mu^p} + \mu^q$$
$$A^{(q,p)} = \frac{1}{qz^{q-1}}\frac{\partial}{\partial z} - \frac{q-1}{2q}\frac{1}{z^q} - z^p \tag{51}$$

For any (p,q) theory with $q > p$ the formula (50) maps it onto the corresponding "dual" theory with $q < p$ and vice versa.

In such way one can easily consider the $(p,1)$ topological theories as dual to the higher critical points of the $(1,p)$ theory with the potential $V_2(x) = \frac{1}{2}x^2$, $W_2 = x$. For this theory the "topological" solution is trivial (for example, the partition function is given by a Gaussian integral and equals to unity) so the basis vectors are

$$f_i^{(1,p)}(z) = z^{i-1} \tag{52}$$

and the Kac-Schwarz operator

$$A^{(1,p)} = \frac{\partial}{\partial z} - z^p \tag{53}$$

preserves reduction of the corresponding $(p,1)$ model in a trivial way

$$A^{(1,p)} f_i^{(1,p)}(z) = [\frac{\partial}{\partial z} - z^p] z^{i-1} =$$
$$= -z^{i+p-1} + (i-1)z^{i-2} = -f_{i+p-1}^{(1,p)}(z) + (i-1)f_{i-1}^{(1,p)}(z) \tag{54}$$

In this particular case we see how the duality formula turns the problem of finding nontrivial basis of [19, 3, 4, 9] to the trivial basis in the Grassmannian (52), corresponding to sphere.

In general case, we have no more the situation when a non-trivial problem reduces to a trivial one. Moreover, it can be easily proven that for a generic (p,q) model the string equation reduces to a sort of higher hypergeometrical equation giving rise to (linear combinations of) generalized hypergeometric functions [21, 22] (for integral formulas and connection to free-field representation see also [23] and references therein).

Indeed, we can obtain some particular solutions of the conditions (36) concerning only the shift $T_{p+q} \to T_{p+q} - t_{p+q}$ as follows. Let us consider the (p,q) model with $q = pn + \alpha$, $\alpha = 1, ..., p-1$; $n = 1, 2, ...$ Using condition of p-reduction we can choose the whole basis in the form

$$\phi_{i+pk} = \mu^{pk}\phi_i \ , \ i = 1, \dots \ , p \tag{55}$$

and therefore eq.(36) give the system of equations for first p vectors:

$$A^{(p,q)}\phi_i = \phi_{i+pn+\alpha} = \mu^{pn}\phi_{i+\alpha} \ , \ i = 1, \dots \ , p \tag{56}$$

where in the case under consideration

$$A^{(p,q)} = \frac{1}{p\mu^{p-1}}\frac{\partial}{\partial\mu} - \frac{p-1}{2p}\frac{1}{\mu^p} + \mu^{pn+\alpha} \equiv$$

$$\equiv N^{(p,q)}(\mu)\frac{1}{p\mu^{p-1}}\frac{\partial}{\partial\mu}[N^{(p,q)}(\mu)]^{-1} \tag{57}$$

and

$$N^{(p,q)}(\mu) = [p\mu^{p-1}]^{1/2}\exp\left(-\frac{p}{p+q}\mu^{p+q}\right)$$

$$q = pn + \alpha \tag{58}$$

After the substitution

$$\phi_i = \mu^{i-1}N^{(p,q)}(\mu)u_i(\mu) \tag{59}$$

and changing the spectral parameter

$$z = \frac{p}{p+q}\mu^{p+q} \tag{60}$$

the system (57) acquires the remarkably simple form

$$A_i u_i = u_{i+\alpha} \ , \ i = 1, \dots \ , p; \ u_{j+p} \equiv u_j \tag{61}$$

with

$$A_i = \frac{d}{dz} + \frac{i-1}{p+q}\frac{1}{z} \ . \tag{62}$$

Equations (61) can be easily solved, say, by series expansion

$$u_i(z) = \sum_{j=0}^{\infty} u_{ij}z^j \tag{63}$$

giving

$$u_{1j} = \prod_{k=0}^{p-1}[j + (\frac{1}{p+q} - 1)k]^{-1}u_{1,j-p} \tag{64}$$

which determines function $u_1(z)$ (up to p arbitrary constants) and others can be obtained by action of (62). Up to these constants a particular solution will be determined by the following formula

$$u_{1,np} = \prod_{l=0}^{n}\prod_{k=0}^{p-1}(pl + (\frac{1}{p+q} - 1)k)^{-1} \tag{65}$$

from which follows that the solutions can be expressed through generic hypergeometric functions ${}_rF_s(z)$. We are going to return to this problem in a separate publication [25].

As a concrete example of the relation to the Hypergeometric equation, we present here the solution to $(2, 2k-1)$ model

$$u_1(\mu) = exp(-\frac{2}{2k+1}\mu^{2k+1})_1F_1(\frac{1}{4k+2}, \frac{1}{2k+1}; \frac{4}{2k+1}\mu^{2k+1}) \tag{66}$$

For the particular case of $k = 1$ this reduces to the well-known solution of the Kontsevich model [8] being a linear combination of the Airy functions [3, 4, 9].

Now, let us only finish with a remark, that formulas (61) are practically equivalent for all $(p, q = np+\alpha)$ series with different n and α. They give a manifestation for a certain cyclic $\mathbf{Z}_p$-symmetry. The only difference in solution to different (p, q)-string equations is the exact order established within the multiplets of order p. The exact sense of this symmetry deserves further investigation.

6 Remarks on $c \to 1$ limit

Let us now make some comments on $c = 1$ situation. From basic point of view we need in generic situation to get the most general (unreduced) KP or Toda-lattice tau-function satisfying some (unreduced) string equation. In a sense this is not a limiting case for $c < 1$ situation but rather a sort of "direct sum" for all (p,q) models. This reflects that in conformal theory coupled to $2d$ gravity there is no more a big difference between $c < 1$ and $c = 1$ situations - quite different before this coupling.

However, there are several particular cases when one can construct a sort of direct $c \to 1$ limit and which should correspond to certain highly "degenerate" $c = 1$ theories. From the general point of view presented above these are nothing but very specific cases of (p, q) string equations, and they could correspond only to a certain very reduced subsector of $c = 1$ theory.

Indeed, it is easy to see, that for two special cases $p = \pm q$ the equations (61) can be simplified drastically, actually giving rise to a single equation instead of a system of them. Of course, these two cases don't correspond to minimal series where one needs (p, q) being coprime numbers. However, we still can fulfill both reduction and Kac-Schwarz condition and these solutions to our equations using naively the formula for the central charge, one mightidentify with $c = 1$ for $p = q$ and $c = 25$ for $p = -q$.

Now, the simplest "topological" theories should be again with $q = 1$. For such case "$c = 1$" turns to be equivalent to a discrete matrix model [5] while "$c = 25$" is exactly what one would expect from generalization of topological Kontsevich-Penner approach [7, 24] (see also R.Dijkgraaf's contribution to this volume). Indeed, taking *non-polynomial* finctions, like

$$W(x) = x^{-\beta}$$

$$Q(x) = x^{\beta} \tag{67}$$

the action would acquire a logariphmic term

$$S_{-\beta,\beta} = -\beta log x + \frac{x^{\beta}}{\mu^{\beta}} \tag{68}$$

while equations (61) give rise just to rational solutions. It is very easy to see that $\beta = 1$ immediately gives "Kontsevich-Penner" result, which rather corresponds to "dual" to $c = 1$ situation with matter central charge being $c_{matter} = 25$ with a highly non-unitary realization of conformal matter.

On the other hand, $p = q = 1$ solution is nothing but a trivial theory, which however becomes a nontrivial discrete matrix model for unfrozen zero-time. Moreover, these particular $p = \pm q$ solutions become nontrivial only if one considers the Toda-lattice picture with negative times being involved into dynamics of the effective theory. On the contrary, we know that $c < 1$ (p, q)-solutions in a sense trivially depends on negative times with the last ones playing the role of symmetry of string equation [5]. It means, that we don't yet understand enough the role of zero and negative times in the Toda-lattice formulation.

7 Conclusion

Let us make some conclusive remarks. We tried to present in the paper the exact mechanism of transitions among different (p, q) solutions of non-perturbative $2d$ gravity in the framework of general scheme proposed in papers [3, 4, 5, 6, 7]. We demonstrated that a naive analytic continuation in the space of Miwa parameters though correct formally leads to certain practical difficulties in explicit description of higher critical points even in trivial topological situation. Instead, we demonstrated a concrete scheme, which allow one to shift "classical" counterparts of the KP times, determined by the coefficients of the potential and by the choice of right variable.

The corresponding integral representation is a direct consequence of the action principle and can be interpreted as a certain field theory integral with a highly nontrivial measure. It obeys manifest $p - q$ symmetry which is evident and restores equivalence in motion along naively two different p- and q-directions. Moreover, the appearance of higher degrees of polynomials can be obtained by transformation from the higher critical points of lower p models.

There is still a lot of open questions. Even in a dual to topological $(p, 1)$ series model there exists nontriviality after $\alpha log X$ term (and negative times terms) are added to the potential. For the $p = 1$ model this gives rise to a separate interesting problem – the discrete Hermitean matrix model [5] and the question is about interpretation of such generalizations of nontrivial theories.

The other question is more deep understanding of generic $c = 1$ situation (which is not reduced to particular "degenerate" cases considered in sect.6) and the role of negative times: symmetry between positive and negative times, the "dissappearing" of negative times in $c < 1$ case etc. Naively, the duality formula leads to a Fredholm equation on basis vector, which can be solved by using the Hermit polynomials, giving rise to the trivialized situation of a discrete 1-matrix model. It is also quite interesting to study the quasiclassical limit of general (p, q) solutions and to compare them with topological theories. This might shed light to the underlying topological structure of naively non-topological theories.

All these problems deserves further investigation and we are going to return to them elsewhere.

We are deeply indebted to R.Dijkgraaf, L.Girardello, A.Lossev, A.Mironov and A.Zabrodin for illuminating discussions. A.M. is grateful to the organizers of the International Workshop on String Theory, Quantum Gravity and the Unification of Fundamental Interactions for warm hospitality in Rome.

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